ON THE STABILITY OF ROTATION OF A RIGID BODY WITH ONE POINT FIXED IN THE EULER CASE

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The differential equations of motion of a rigid body with one point fixed in the Euler case are as follows:

$$A\frac{dp}{dt} - (B - C)\,qr = 0, \ldots, \qquad \frac{d\gamma_1}{dt} = r\gamma_2 - q\gamma_3, \ldots \tag{1}$$

where A, B, C, are the moments of inertia of the body about the principal axes of the central ellipsoid; p, q, r, are the components of the instantaneous angular velocity along the moving coordinate axes x, y, z, which coincide with the principal axes of inertia; γ_1, γ_2, γ_3, are the direction cosines of a fixed axis ζ with respect to the moving coordinate system xyz.

Among all possible motions of a rigid body in the Euler case, the most important from the practical point of view are constant rotations about principal axes of inertia passing through the fixed point and coinciding with the ζ axis.

Many authors have investigated the stability of this kind of rotations. For example, the very well-known purely geometrical solution of this problem by the Poinsot method is to be found in any full course of mechanics. Nekrasov has investigated the stability of these rotations by using the Liapunov first approximation (see his course of theoretical mechanics), whereas Chetaev [2] has used the Liapunov direct method. In the above investigations the problem of stability was solved only with respect to the three variables p, q, r. In this paper the problem of unconditional stability of particular solutions of the system (1) with respect to all variables of the problem, p, q, r, γ_1, γ_2, γ_3, is solved.

We will first consider a rotation about the x axis of the central ellipsoid of inertia expressed by the following particular solution of the system (1):

$$p = \omega = \text{const}, \quad q = 0, \quad r = 0, \quad \gamma_1 = 1, \ \gamma_2 = 0, \quad \gamma_3 = 0$$

The above solution describes the motion of a holonomic conservative system. Chetaev, using the Liapunov direct method, demonstrated [1] that such a motion is stable only when the system of equations of the perturbed motion has an integral that is sign-definite for variations of the variables of the problem.

In the Euler case the following first integrals are known:

$$A^2p^2 + B^2q^2 + C^2r^2 = \text{const} = H^2, \qquad Ap^2 + Bq^2 + Cr^2 = \text{const} = h$$
$$Ap\gamma_1 + Bq\gamma_2 + Cr\gamma_3 = \text{const} = k, \qquad \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = \text{const} = 1$$

The variations of the variables in the perturbed motion will be denoted

$$p = \omega + \alpha, \quad q = \beta, \quad r = \delta, \quad \gamma_1 = 1 + \xi_1, \quad \gamma_2 = \xi_2, \quad \gamma_3 = \xi_3$$

The variational equations

$$A\frac{d\alpha}{dt} - (B - C)\beta\delta = 0, \ldots \qquad \frac{d\xi_1}{dt} = \delta\xi_2 - \beta\xi_3, \ldots$$

have the following first integrals

$$\begin{aligned} V_1 &= A^2\alpha^2 + B^2\beta^2 + C^2\delta^2 + 2A^2\omega\alpha = \text{const} \\ V_2 &= A\alpha^2 + B\beta^2 + C\delta^2 + 2A\omega\alpha = \text{const} \\ V_3 &= A\alpha\gamma_1 + B\beta\gamma_2 + C\delta\gamma_3 + A(\alpha + \omega\xi_1) = \text{const} \\ V_4 &= \xi_1^2 + \xi_2^2 + \xi_3^2 + 2\xi_1 = 0 \end{aligned} \tag{2}$$

Moreover, from V_4 we obtain the equality

$$2\xi_1 = -\xi_1^2 - \xi_2^2 - \xi_3^2 \tag{3}$$

We will construct the Liapunov function in the form of the following combination of integrals:

$$V = V_1 + V_2^2 - 2A\omega V_3 + A^2\omega^2 V_4 \tag{4}$$

Substituting (2) in (4), and grouping the terms appropriately, we obtain

$$V = A^2(\alpha - \omega\xi_1)^2 + (B\beta - A\omega\xi_2)^2 + (C\delta - A\omega\xi_3)^2 + (A\alpha^2 + B\beta^2 + C\delta^2 + 2A\omega\alpha)^2 \tag{5}$$

If it could be shown that the function V vanishes only when all variables in it are zero, then the definiteness (positive) of V would be proved. We will first examine those values of the variables which make the first three expressions in parentheses equal to zero in (5) and are not all zero simultaneously:

$$\alpha = \omega\xi_1, \quad \beta = \frac{A}{B}\omega\xi_2, \quad \delta = \frac{A}{C}\omega\xi_3, \quad \xi_1 \neq 0, \quad \xi_2 \neq 0, \quad \xi_3 \neq 0 \tag{6}$$

The function V will contain only the fourth expression in parentheses, which by using (3) can be transformed into

$$V = A^2\omega^4\left[\left(\frac{A}{B}-1\right)\xi_2^2 + \left(\frac{A}{C}-1\right)\xi_3^2\right]^2$$

On the strength of (3) it is seen that the above expressions cannot be made zero when the values of the variables as given in (6) are sufficiently small and when the moments of inertia A, B, C satisfy the inequality

$$(A-B)(A-C) > 0 \qquad (*)$$

On the other hand, it is easy to show, that the function V can be made zero for the following values of the variables:

$$\xi_2 = \xi_3 = \beta = \delta = 0, \qquad \xi_1 \neq 0, \qquad \alpha = \omega\xi_1 \qquad (7)$$

We will show, however, that this last case can be excluded if the initial perturbations, α_0, β_0, ξ_{10}, ξ_{20}, ξ_{30}, are sufficiently small and do not make the integrals V_1, V_2, V_3, simultaneously zero. Indeed, if we take into account (3), then the equalities (7) become

$$\xi_2 = \xi_3 = \beta = \delta = 0, \qquad \xi_1 = -2, \qquad \alpha = -2\omega$$

which substituted into (2) make V_1, V_2, V_3, equal to zero simultaneously.

This means that sufficiently small perturbations will not make the integrals (2) equal to zero simultaneously if

$$\alpha_0^2 + \beta_0^2 + \delta_0^2 < 4\omega^2, \qquad \xi_{10}^2 + \xi_{20}^2 + \xi_{30}^2 < 4$$

and then the function V will be positive-definite. On the strength of the well-known Liapunov theorem [3] we conclude that the constant rotation about the smallest ($A > B \geqslant C$) and the largest ($A < B \leqslant C$) semiaxes of the ellipsoid of inertia is stable with respect to all the variables of the problem, p, q, r, γ_1, γ_2, γ_3. This conclusion can also be reached on the strength of the perturbed equation $V' \equiv 0$.

A constant rotation of the body about the intermediate semiaxis of the ellipsoid of inertia corresponds to the following particular solution of the system (1):

$$p = 0, \quad q = \omega = \text{const}, \quad r = 0, \quad \gamma_1 = 0, \quad \gamma_2 = 1, \quad \gamma_3 = 0 \qquad (8)$$

and it proves to be unstable. Using the same method as before, for the variations of the perturbed motion we will have the following function:

$$V = (A\alpha - B\omega\xi_1)^2 + B^2(\beta - \omega\xi_2)^2 + (C\delta - B\omega\xi_3)^2 + (A\alpha^2 + B\beta^2 + C\delta^2 + 2B\omega\beta)^2$$

When the variations assume the values

$$\alpha = \frac{B}{A}\omega\xi_1, \quad \beta = \omega\xi_2, \quad \delta = \frac{B}{C}\omega\xi_3, \quad \xi_1 \neq 0, \quad \xi_2 \neq 0, \quad \xi_3 \neq 0 \qquad (9)$$

the first three expressions in parentheses in V vanish and V becomes

$$V = B^2\omega^4\left[\left(\frac{B}{A}-1\right)\xi_1^2+\left(\frac{B}{C}-1\right)\xi_3^2\right]^2$$

which combined with the inequality (*), can be made equal to zero when variations have the values given in (9) and when ξ_1, ξ_3, satisfy the equality

$$C(A-B)\xi_1^2 - A(B-C)\xi_3^2 = 0$$

It thus follows from the above that in this last case the function V is not definite and the Liapunov conditions for stability are not satisfied.

In order to prove the instability of the particular solution (8) we will consider the function

$$V = \alpha\delta + \xi_1\xi_3$$

and in the region $V > 0$ select a strip where the variations α, β, ξ_1, ξ_3 have the values (9). In view of the equations of perturbed motion

$$A\frac{d\alpha}{dt} - (B-C)(\omega+\beta)\delta = 0, \ldots \qquad \frac{d\xi_1}{dt} = \delta(1+\xi_2) - \xi_3(\omega+\beta), \ldots \qquad (10)$$

and after the substitution of (9) in the expression for V, the derivative of V will be

$$V' = (\omega+\beta)\left[\frac{A-B}{C}\alpha^2 + \frac{B-C}{A}\delta^2 + \frac{A-B}{A}\xi_1^2 + \frac{B-C}{C}\xi_3^2\right]$$

If $(\omega+\beta) = 0$, then the instability follows from (10). If $(\omega+\beta) > 0$, then V will be positive-definite in the region $V > 0$. The function V does not depend explicitly on t, and therefore allows an infinitely small upper bound. Now, by one of the Chetaev theorems [2] on instability, we can conclude that the rotation of a rigid body about the intermediate semiaxis of the ellipsoid of inertia is unstable.

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